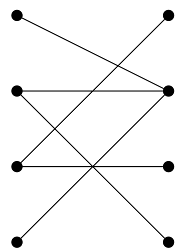
Chapter 2 - Matchings and Coverings

Let $G = (V, E)$ be a graph.

- A **matching** M is a subset of E such that each vertex in V is incident with at most one edge in M.
- A vertex incident with an edge in M is called **matched** (or **covered**), otherwise it is **unmatched** (or **exposed**).
- A matching is **perfect** if every vertex is matched.
- a k-factor in G is an induced k-regular subgraph.

1: Find a maximum matching in the following graph.



Let $G = (V, E)$ and H be a graphs.

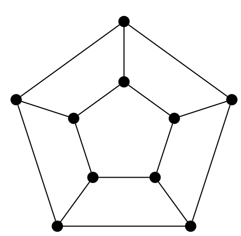
- A **packing** of G in H is a set of vertex subgraphs each isomorphic to H. (Copies don't need to be induced)
- A **covering** is $U \subseteq V$ such that each copy of H in G contains a vertex in U.

Typical problem is minimize covering and maximize packing.

2: Show that smallest covering is at least as large as the largest packing.

Solution: If we have vertex disjoint copies, easily you need at least that many vertices.

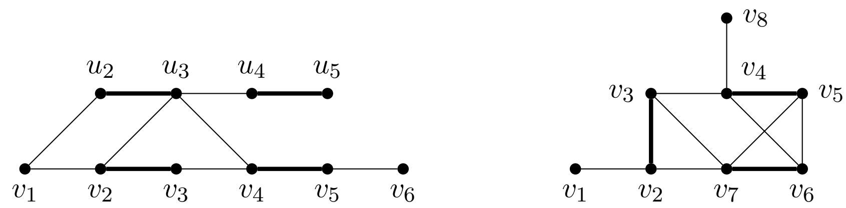
3: Find a largest matching. Find smallest covering for matching.



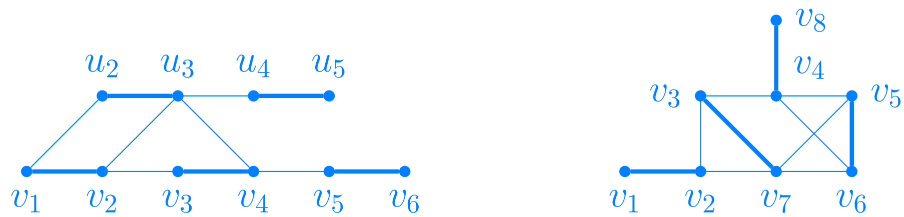
Solution:

A path P is M**-alternating** if $E(P) \setminus M$ is a matching. An M**-alternating** path P is M**-augmenting** if P has positive length and both its endpoints are unmatched/exposed in M. Augmented $M' = M \Delta E(P)$.

4: Assume there is a matching M (thick lines). Find M-augmenting path(s) and augment M.



Solution: An M-augmenting path is a path P with endpoints exposed, inner points covered and the edges of the matching are alternating on P. On the left for example $v_1, v_2, v_3, v_4, v_5, v_6$. On the right it is $v_1, v_2, v_3, v_7, v_6, v_5, v_4, v_8$.



5: Can we use augmenting walks instead of paths? It particular, examine walk $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_5, v_4, v_8$ in the graph on the right-hand side.

Solution: We cannot augment on it. Both v_4 and v_5 would have two matching edges.

Theorem (Berge 1957) Let G be a graph with a matching M. Then M is maximum if and only if G has no M-augmenting path.

6: Prove Theorem. (Hint: symmetric difference)

Solution: $\Rightarrow$: Augmenting path increases the size of the matching
$\Leftarrow$. Consider M' being a matching with more edges than M. Take the symmetric difference of M and M'. It is a graph of maximum degree two, which gives set of even cycles and paths. Implies one of the paths must be M-augmenting.

Theorem 2.1.1 (König 1931) Let G be a bipartite graph. The cardinality of maximum matching is equal to the minimum vertex cover of its edges.

Proof One direction is clear. Take maximum matching M and construct a nice cover. Let the bipartition be $A \cup B = V(G)$. For each edge $ab \in M$, we put to the cover U vertex b iff it is an endpoint of some augmenting path that starts in A. Otherwise we put a to U.

7: Let $ab \in E$. Which cases one needs to consider to argue ab has at least one endpoint in U?

Solution: If $ab \in M$, a or b in U. At least one of a, b covered M, otherwise M not maximal.

If a is not covered by M, then b is covered and ab is an M-alternating path, meaning $b \in U$.
Hence a is covered and part of a matching edge $ab' \in M$. If $a \in U$, we are done. Hence b' in U and there is an M-alternating path P ending in b'. The path P either uses b and then b is also covered and $b \in M$ or P does not use b, then the path can be extended as $Pb'ab$. In either case, by maximality of M, b is matched in M. And M-alternating path ends there, hence $b \in U$. Why the path argument works? Discuss why, how can we do $Pb'ab$? Will it be a path?

Theorem (Hall 1935) Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$. G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

8: Show that the condition of Hall's theorem is necessary. I.e. if there is a matching M matching every vertex in A, then $|N(S)| \geq |S|$ for all $S \subseteq A$. This condition is sometimes called marriage condition or Hall's condition.

Solution: It is obvious but good to realize the condition is necessary. Each vertex in S is matched by M to a different vertex in $N(S)$, hence $|N(S)| \geq |S|$.

9: Prove Hall's theorem by finding M-augmenting path. Let $a \in A$ be unmatched in M. Consider $A' \subseteq A$ that can be reached from a by an M-alternating path. Use the marriage condition (or Hall's condition) on A'.

Solution: First we observe that all vertices in $A' \setminus \{a\}$ are matched. That is how we reached them. Now $|N(A)| \geq |A|$, so there must be an unmatched $b \in |N(A)|$. Say $vb \in E$. Then there is M-alternating path P from a to v and Pb is an M-augmenting path. Draw a picture to see how the augmenting paths behave.

10: Prove Hall's theorem by induction on $|A|$. Base case. Then try to resolve case $|N(S)| \geq |S| + 1$ for every proper subset. Then use a proper subset satisfying $|N(S)| = |S|$.

Solution: Base case is $|A| = 1$ and Hall's condition is saying there is at least one neighbor to match.

If $|N(S)| \geq |S| + 1$ for every proper $S \subset A$, then pick any edge ab, and consider $G - ab$. Notice that Hall's condition is still true because $N(S)$ maybe lost b but that is all. From induction we get a matching M and add edge ab to it.

Finally, let $A' \subset A$ be such that $|A'| = |N(A')|$. Let $N(A') = B'$. By induction, we can find a matching in $G' := G[A', B']$. We also need to find a matching in $G - A' - B'$. We need to check Hall's condition for $G - A' - B'$. Let $S \subseteq A - A'$. Suppose for contradiction that $|N(S) - B'| < |S|$. But then $S \cup A$ gives $N(S \cup A)| = |(N(S) \setminus N(A)) \cup N(A)| = |N(S) - B'| + |N(A)| < |S| + |A|$, which is a contradiction. Hence there is a matching in $G - A' - B'$ for $A - A'$.

11: Prove Hall's theorem by considering a subgraph H of G that is edge-minimal while satisfying Hall's condition. Clearly, $d_H(a) \geq 1$ for each $a \in A$. The goal is to show $d_H(a) = 1$ for each $a \in A$, that means H is a matching for A.

Solution: Let a have neighbors $b_1, b_2 \in B$. $H - b_1$ and $H - b_2$ violate Hall's condition by minimality. Hence Exist $A_1, A_2 \subseteq A$ such that $B_i := N_{H-ab_i}(A_1)$ and $|B_i| < |A_i|$. Notice $b_1 \in B_2$ and $b_2 \in B_1$.
$|N_H(A_1 \cap A_2 \setminus \{a\})| = |B_1 \cap B_2| = |B_1| + |B_2| - |B_1 \cup B_2| \leq |A_1| - 1 + |A_2| - 1 + |A_1 \cup A_2|$
$= |A_1 \cap A_2| - 2 = |A_1 \cap A_2 \setminus \{a\}| - 1$

A matching is **perfect** if it is 1-factor.

12: Show that every k-regular ($k \geq 1$) bipartite graph has a 1-factor (means perfect matching).

Solution: Verify Hall's condition. Let $S \subseteq A$. The number x of edges leaving S is $x = k|S|$. From the other side, $x \leq k|N(S)|$. Together we get $|S| \leq |N(S)|$.

Let $S_1, S_2, \ldots, S_n$ be nonempty finite sets. Then this collection of sets has a **system of distinct representatives** if there exist n distinct elements $x_1, x_2, \ldots, x_n$ such that $x_i \in S_i$ for $1 \leq i \leq n$.

13: Find a system of distinct representatives for the following sets

$$S_1 = \{1,2,3\} \quad S_2 = \{2,4,6\} \quad S_3 = \{2,5,6\} \quad S_4 = \{3,4,5\} \quad S_5 = \{1,4,6\}$$

Solution: Almost greedy algorithm will work.

Theorem (Original formulation of Hall's Theorem)
A collection $\{S_1, S_2, \ldots, S_n\}$ of nonempty finite sets has a system of distinct representatives if and only if for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.

14: Use Hall's theorem to prove its original formulation.

Solution: Turn the system of distinct representatives into a bipartite graph, when vertices of one part correspond to elements in $\cup_i S_i$ and vertices the other part correspond to sets $S_1, \ldots, S_n$.